

Determination of Optimal Product Mix for Profit Maximization using Linear Programming

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ABSTRACT

This paper demonstrates the use of linear programming methods in order to determine the optimal product mix for profit maximization. There had been several papers written to demonstrate the use of linear programming in finding the optimal product mix in various organization. This paper is aimed to show the generic approach to be taken to find the optimal product mix.

Keywords - Basic solution, Degeneracy, Linear Programming, Objective function, Simplex Method, Slack and Surplus variables.

I. INTRODUCTION

A programming problem is a class of problems that determines the optimal allocation of limited resources to meet given objectives. The resources may be men, material, machine and land. (Iheagwara et al 2014). Linear Programming (LP) can be defined as a mathematical technique for determining the best allocation of a firm's limited resources to achieve optimal goal. (Yahya et al 2012). Linear Programming (LP) is a mathematical technique to achieve profit maximization or cost minimization in a mathematical model whose requirements are represented by linear relationship. A linear programming problem may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints. Linear Programming, an operations research technique is widely used in finding solutions to complex managerial decision problems. (Ezema et al, 2012)

In a multi-product company it is often a challenging task to find the appropriate quantities of the product to be produced in order to maximize the profit. It is quite a tough management decision to fix the product quantities under the given constraints of the organization. Linear programming comes to much use in finding the optimal product mix.

II. LINEAR PROGRAMMING PROBLEM

The general linear programming problem can be described as follows:

Given a set of m linear equations or inequalities in r variables, we wish to find non-negative values of this variables which will satisfy the constraints and maximize or minimize some linear function of the variables. (Hadley 2002)

Mathematically, the above statement means, we have m equalities or equations in r variables (m can be greater than, less than or equal to r) of the form:

$$a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + \dots + a_{ir} x_r \{ \geq, =, \leq \} b_i, \quad i = 1, 2, 3, \dots, m \quad (1)$$

where for each constraints one and only one of the signs $\geq, =, \leq$ holds, but the sign may vary from one constraints to another. We seek the values of the variables x_j satisfying (1) and $x_j \geq 0, j = 1, 2, 3, \dots, r,$ (2)

which maximize or minimize a linear function $z = c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_r x_r$ (3)

(a_{ij}, b_i, c_j s are assumed to be known constants) (Hadley 2002)

The function z is to be optimized and it is called the

objective function.

Condition (2) which implies that the variables should be non-negative are called the **non-negativity restrictions**.

Any set of x_j which satisfies (1) is regarded as a **solution** to the linear programming problem whereas any solution which shall satisfy the non-negativity restrictions is regarded as a **feasible solution**. Any feasible solution which shall optimize the objective function is regarded as an **optimal feasible solution**. The main task of solving a linear programming problem is to find an optimal feasible solution

III. SLACK AND SURPLUS VARIABLES:

Since it is much more convenient to work with equations than inequalities it is desirable to convert any inequalities in (1) into equations. The extra variables adding which such conversion is possible are called the slack and surplus variables.

Consider the constraints having \leq sign in (1)

A typical constraint of this category (say constraint g) can be written as

$$a_{g1} x_1 + a_{g2} x_2 + a_{g3} x_3 + \dots + a_{gr} x_r \leq b_g$$

Let us introduce a new variable $x_{(r+g)} \geq 0$, where

$$a_{g1} x_1 + a_{g2} x_2 + a_{g3} x_3 + \dots + a_{gr} x_r + x_{(r+g)} = b_g$$

or $x_{(r+g)} = b_g - (a_{g1} x_1 + a_{g2} x_2 + a_{g3} x_3 + \dots + a_{gr} x_r) \geq 0$

$x_{(r+g)}$ is called a **slack variable**.
 Let us now consider the constraints having \geq sign in (1)

A typical constraint of this category (say constraint k) can be written as

$$a_{k1} x_1 + a_{k2} x_2 + a_{k3} x_3 + \dots + a_{kr} x_r \geq b_k$$

Let us now define a new variable $x_{(r+k)} \geq 0$, where

$$a_{k1} x_1 + a_{k2} x_2 + a_{k3} x_3 + \dots + a_{kr} x_r = b_k + x_{(r+k)}$$

or $x_{(r+k)} = (a_{k1} x_1 + a_{k2} x_2 + a_{k3} x_3 + \dots + a_{kr} x_r) - b_k$

$x_{(r+k)}$ is called a **surplus variable**.

IV. BASIC SOLUTION

Given a system of m simultaneous linear equations in n unknowns, $Ax = b$ ($m < n$) and $\text{rank}(A) = m$. If any $m \times m$ non-singular matrix is chosen from A, and if all the (n-m) variables not associated with the columns of this matrix are set to zero, the solution to the resulting system of equation is called a **basic solution**. The m variables which can be different from zero are called the **basic variables**. (Hadley 2002)

V. DEGENERACY

A basic solution to $Ax = b$ is degenerate if one or more of the basic variables vanish.

VI. SIMPLEX METHOD

In order to determine an optimal basic feasible solution, we might attempt to find all basic solutions (including those which are not feasible) and the select one which shall yield the optimal value of the objective function. Since there is only finite number of basic solutions, in theory, this is very much possible. But since the number of basic solutions increases very rapidly as the number of variables increases, the above procedure is not very efficient. Simplex method sorts out this problem. Simplex method is an algebraic iterative method that proceeds in a systematic way from an initial basic feasible solution to other basic feasible solutions and ultimately reaching the optimal basic feasible solution in a finite number of steps. This method was developed by George Dantzig in 1947.

In order to derive solutions for the linear programming problem using Simplex method, the objective function and the constraints must be standardized.

The characteristics of the standard form are:

- All the constraints must be expressed in the form of equations except the non-negativity

constraints which remains in the form of inequalities.

- The right-hand side of each constraints equation must be non-negative, that is, the b_i s must be non-negative. Since from strictly theoretical point of view the each b_i can have any sign we shall first ensure that each of the $b_i \geq 0$. In order to ensure this we simply just shall multiply both sides of the inequality by (-1) whenever we shall find the corresponding b_i to be negative in the original inequality. This would make the corresponding b_i non-negative and of course shall alter the direction of the inequality.
- The **TABLE 1** format can be used to solve the linear programming problem using Simplex method.

VII. THE SIMPLEX ALGORITHM

Step 1: Examine the $(z_j - c_j)$

Case 1: If $(z_j - c_j) \geq 0$. In this case the given basic solution is optimal.

Case 2: One or more $(z_j - c_j) < 0$ and for at least one a_k for which $(z_k - c_k) < 0$, all $y_{ik} \leq 0$. Then there is an unbounded solution.

Case 3: One or more $(z_j - c_j) < 0$ and each of these has $y_{ij} > 0$ for at least one i. Select any of the vectors a_k to enter the basis.

Step 2: When Case 3 holds true, compute the vector to be removed from the basis matrix using

$$(x_{Br} / y_{rk}) = \min\{ (x_{Bi} / y_{ik}), y_{ik} > 0 \} = \theta$$

Column r is removed and replaced by a_k .

Step 3: Using the below listed formulas compute the new values for all j. This shall yield the quantities of interest for the new basic feasible solution. Return to Step 1.

$$\bar{x}_{Bi} = [x_{Bi} - x_{Br} (y_{ik} / y_{rk})] \quad (i \neq r) \quad (i = 1, 2, 3, \dots, m) \dots \dots [4-1]$$

$$\bar{x}_{Br} = (x_{Br} / y_{rk}) \dots \dots [4-2]$$

$$\bar{z} = z + (x_{Br} / y_{rk}) (c_k - z_k) = z + \theta (c_k - z_k) \dots \dots [4-3]$$

$$[y_{ij} - y_{rj} (y_{ik} / y_{rk})] = y_{ij} \quad (i = 1, 2, \dots, m) \text{ and } (i \neq r) \dots \dots [4-4]$$

$$y_{rj} = (y_{rj} / y_{rk}) \dots [4-5]$$

$$\bar{z}_j - c_j = [z_j - c_j + (y_{rj} / y_{rk}) (c_k - z_k)] \dots \dots [4-6]$$

In absence of degeneracy, this iterative procedure leads to an optimal basic solution in finite number of steps.

VIII. THE PROBLEM:

Assume a certain manufacturing firm produces 'n' number of products, namely, $P_1, P_2, P_3, \dots, P_n$.

Assume there are 'm' raw materials required to manufactures these products. We can name them as $R_1, R_2, R_3, \dots, R_m$. Now refer **TABLE 2**. It lists the total quantity of each raw material available.

TABLE 3 lists the amount of each raw material required to produce a crate of each of the product listed.

TABLE 4 lists the average cost price and selling price of a crate of each product. Under the given scenario, we want to find the optimal product mix that shall maximize the profit.

IX. MODEL FORMULATION

X_i is the quantity of product P_i ($1 \leq i \leq n$)
 $Z = (\text{pr}_1) X_1 + (\text{pr}_2) X_2 + (\text{pr}_3) X_3 + \dots + (\text{pr}_n) X_n$

Subject to

$$y_{11} X_1 + y_{21} X_2 + y_{31} X_3 + \dots + y_{n1} X_n \leq Q_1$$

$$y_{12} X_1 + y_{22} X_2 + y_{32} X_3 + \dots + y_{n2} X_n \leq Q_2$$

.....

$$y_{1m} X_1 + y_{2m} X_2 + y_{3m} X_3 + \dots + y_{nm} X_n \leq Q_m$$

$$X_i \geq 0 \text{ for } i = 1, 2, 3, \dots, n$$

Now introducing slack variables to convert inequalities to equation we get

$$Z = (\text{pr}_1) X_1 + (\text{pr}_2) X_2 + (\text{pr}_3) X_3 + \dots + (\text{pr}_n) X_n$$

$$y_{11} X_1 + y_{21} X_2 + y_{31} X_3 + \dots + y_{n1} X_n + X_{(n+1)} = Q_1$$

$$y_{12} X_1 + y_{22} X_2 + y_{32} X_3 + \dots + y_{n2} X_n + X_{(n+2)} = Q_2$$

.....

$$y_{1m} X_1 + y_{2m} X_2 + y_{3m} X_3 + \dots + y_{nm} X_n + X_{(n+m)} = Q_m$$

$$X_i \geq 0 \text{ for } i = 1, 2, 3, \dots, n, (n+1), \dots, (n+m)$$

Using suitable software like excel solver we can get the necessary values of the variables as shown in **TABLE 5.**

X. CONCLUSION

Based on the given data, the optimum results as derived shows that the quantity of product P_i is v_i and the maximum value of profit is z_0 . Mentionworthy, the quantity of certain product to be produced under optimal condition can be zero indicating that the manufacturing firm should stop the production of that particular product or products in order to maximize profit.

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Table 1: Simplex Table

			c_1	c_2	c_n	(+/-) M	(+/-) M
c_B	Vectors in basis	b	a_1	a_2	a_n	q_1	q_s
c_{B1}	b_1	x_{B1}	y_{11}	y_{12}	y_{1n}	$y_{1(n+1)}$		$y_{1(n+s)}$
c_{B2}	b_2	x_{B2}	y_{21}	y_{22}	y_{2n}	$y_{2(n+1)}$		$y_{2(n+s)}$
.....								
.....								
.....								
c_{Bm}	b_m	x_{Bm}	y_{m1}	y_{m2}	y_{mn}	$y_{m(n+1)}$		$y_{m(n+s)}$
		$Z =$	$z_1 - c_1 =$	$z_2 - c_2 =$	$z_n - c_n =$

Table 2: Quantity of raw materials available in stock

Raw Materials	Quantity available
R_1	Q_1
R_2	Q_2
R_3	Q_3
R_m	Q_m

Table 3: Quantity of raw materials needed to produce a crate of each product

Products	R_1	R_2		$R_{(m-1)}$	R_m
P_1	y_{11}	y_{12}		$y_{1(m-1)}$	y_{1m}
P_2	y_{21}	y_{22}		$y_{2(m-1)}$	y_{2m}
P_3	y_{31}	y_{32}		$y_{3(m-1)}$	y_{3m}
P_n	y_{n1}	y_{n2}		$y_{n(m-1)}$	y_{nm}

Table 4: Average cost and selling price of a crate of each product

Products	Average cost price	Average Selling Price	Profit
P_1	c_1	s_1	pr_1
P_2	c_2	s_2	pr_2
P_3	c_3	s_3	pr_3
P_n	c_n	s_n	pr_n

Table 5: Variables and their values

Variables	Values
X_1	V_1
X_2	V_2
X_3	V_3
X_n	V_n

$$Z = z_0 = z_{\max}$$